

## A note on holonomic constraints

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In honour of my friend John Stachel.

### 1. Introduction.

This note is a preliminary account of research undertaken jointly with G. Marmo of Napoli and P. Urbanski of Warsaw.

We propose a new description of dynamics of autonomous mechanical systems which includes the momentum-velocity relation. This description is formulated as a variational principle of virtual action more complete than the Hamilton Principle. The inclusion of constraints in this description is the main topic of the present note. We give examples and models of constraints in variational formulations of statics and dynamics of autonomous systems.

A complete description of the dynamics of a mechanical system involves both external forces and momenta. In a fixed time interval the dynamics is a relation between the motion of the system in configuration space, external forces applied to the system during the time interval, and the initial and final momenta. This relation is derived from a variational principle which involves variations of the end points. Constrained systems are idealized representations of unconstrained systems. Such idealizations are appropriate when forces at our disposal are unable to move the configuration of the system away from a subset of the configuration space by a perceptible distance. This description fits at least the case of holonomic constraints. We believe that constraints should be imposed on virtual displacements. Holonomic constraints are usually interpreted as restrictions on configurations of a mechanical system. Nonholonomic constraints are additional restrictions imposed on velocities. This traditional terminology is not adapted to our concept of constraints as imposed on virtual displacements and only indirectly affecting configurations and velocities. Our concept of nonholonomic constraints makes perfect sense for static systems even if velocities do not appear in the description of such systems. We will use the terms *configuration constraints* and *velocity constraints* instead of *holonomic* and *nonholonomic* constraints.

### 2. Geometric structures.

Let  $Q$  be the Euclidean affine space of Newtonian mechanics. The model space for  $Q$  is a vector space  $V$  of dimension 3. The Euclidean structure is represented by a metric tensor  $g: V \rightarrow V^*$ . The space  $V^*$  is the dual of the model space. The canonical pairing is a bilinear mapping

$$\langle \cdot, \cdot \rangle: V^* \times V \rightarrow \mathbb{R}. \quad (1)$$

We denote by  $q_1 - q_0$  the vector associated with the points  $q_0$  and  $q_1$ . We write  $q_1 = q_0 + v$  if  $v = q_1 - q_0$ . The norm  $\|v\|$  of a vector  $v \in V$  is defined by

$$\|v\| = \sqrt{\langle g(v), v \rangle}. \quad (2)$$

The derivative of a function  $F: Q \rightarrow \mathbb{R}$  is the mapping

$$DF: Q \times V \rightarrow \mathbb{R}: (q, v) \mapsto \frac{d}{ds} F(q + sv)|_{s=0}. \quad (3)$$

The first and second derivatives of a differentiable curve  $\gamma: \mathbb{R} \rightarrow Q$  are mappings  $\dot{\gamma}: \mathbb{R} \rightarrow V$  and  $\ddot{\gamma}: \mathbb{R} \rightarrow V$ .

The tangent bundle  $\mathsf{T}Q$  is identified with  $Q \times V$ , the cotangent bundle  $\mathsf{T}^*Q$  is identified with  $Q \times V^*$ , the second tangent bundle  $\mathsf{T}^2Q$  is identified with  $Q \times V \times V$ , the iterated tangent bundle  $\mathsf{TT}Q$  is identified with  $Q \times V \times V \times V$ , and the tangent of the cotangent bundle  $\mathsf{TT}^*Q$  is identified with  $Q \times V^* \times V \times V^*$ . We have the projections

$$\tau_Q: \mathsf{T}Q \rightarrow Q: (q, \dot{q}) \mapsto q, \quad (4)$$

$$\tau_{\mathsf{T}Q}: \mathsf{TT}Q \rightarrow \mathsf{T}Q: (q, \dot{q}, \delta q, \delta \dot{q}) \mapsto (q, \dot{q}), \quad (5)$$

$$\mathsf{T}\tau_Q: \mathsf{TT}Q \rightarrow \mathsf{T}Q: (q, \dot{q}, \delta q, \delta \dot{q}) \mapsto (q, \delta q), \quad (6)$$

and the canonical involution

$$\kappa_Q: \mathsf{TT}Q \rightarrow \mathsf{TT}Q: (q, \dot{q}, \delta q, \delta \dot{q}) \mapsto (q, \delta q, \dot{q}, \delta \dot{q}). \quad (7)$$

For each subset  $C$  of  $Q$  we have the *tangent set*

$$\begin{aligned} \mathsf{TC} = \{ (q, \delta q) \in \mathsf{T}Q ; \text{ there is a curve } \gamma: \mathbb{R} \rightarrow Q \\ \text{such that } \gamma(0) = q, D\gamma(0) = \delta q, \text{ and } \gamma(s) \in C \text{ if } s \geq 0 \} \end{aligned} \quad (8)$$

The space

$$\overset{\circ}{\mathsf{T}}Q = Q \times \overset{\circ}{V} = \{ (q, v) \in \mathsf{T}Q; v \neq 0 \} \quad (9)$$

is the tangent bundle with the zero section removed.

### 3. Statics of a material point.

We consider the statics of a material point in the Euclidean affine space  $Q$  of Newtonian physics.

An element  $(q, \delta q)$  of  $\mathsf{T}Q$  is a *virtual displacement* and an element  $(q, f)$  of  $\mathsf{T}^*Q$  represents an *external force*. The evaluation

$$\langle (q, f), (q, \delta q) \rangle = \langle f, \delta q \rangle \quad (10)$$

of an external force  $(q, f) \in \mathsf{T}^*Q$  on a virtual displacement  $(q, \delta q) \in \mathsf{T}Q$  is the *virtual work* performed by an external device controlling the configuration of the system.

*Admissible displacements* form a subset  $C^1 \subset \mathsf{T}Q$ . If  $(q, \delta q)$  is an admissible displacement, then  $(q, k\delta q)$  is again an admissible displacement for each number  $k \geq 0$ . The set  $C^1$  represents constraints imposed on virtual displacements. Implicitly it restricts *admissible configurations* to the set

$$C^0 = \{ q \in Q; (q, \delta q) \in C^1 \text{ for some } \delta q \in V \}. \quad (11)$$

The inclusion  $C^1 \subset \mathsf{TC}^0$  is usually satisfied. We say that constraints are *configuration constraints* if  $C^1 = \mathsf{TC}^0$ . The set  $C = C^0$  itself is called a *configuration constraint*. A *simple two-sided configuration constraint* is an embedded submanifold  $C \subset Q$

There is a function

$$\sigma: C^1 \rightarrow \mathbb{R} \quad (12)$$

assigning to each admissible virtual displacement the virtual work that an external device has to perform in order to effect this displacement. This *virtual work function* is differentiable on  $C^1 \cap \overset{\circ}{\mathsf{T}}Q$  and positive homogeneous in the sense that

$$\sigma(q, k\delta q) = k\sigma(q, \delta q) \quad (13)$$

if  $k \geq 0$ . A typical example of a virtual work function is the mapping

$$\sigma: C^1 \rightarrow \mathbb{R}: (q, \delta q) \mapsto DU(q, \delta q) \quad (14)$$

derived from an *internal energy function*  $U$  defined in a domain large enough to make the derivative  $DU(q, \delta q)$  meaningful. In the case of a configuration constraint  $C \subset Q$  it is enough to have the internal energy defined on  $C$ . The function

$$\sigma: C^1 \rightarrow \mathbb{R}: (q, \delta q) \mapsto \rho(q) \|\delta q\| \quad (15)$$

represents virtual work due to friction.

The response of the system to external control is represented by a set  $S \subset \mathbb{T}^*Q$  of external forces satisfying the *principle of virtual work*

$$\langle f, \delta q \rangle \leq \sigma(q, \delta q) \text{ for each virtual displacement } (q, \delta q) \in C^1. \quad (16)$$

The set  $S$  is the *constitutive set* of the system. It can be viewed as the list of possible configurations of the system together with external forces compatible with these configurations. If  $\sigma(q, -\delta q) = -\sigma(q, \delta q)$  and the constraints are two-sided, then the principle of virtual work assumes the simpler form

$$\langle f, \delta q \rangle = \sigma(q, \delta q) \text{ for each virtual displacement } (q, \delta q) \in C^1. \quad (17)$$

EXAMPLE 1. Let a material point be constrained to a circular hoop with the center at  $q_0 \in Q$  and radius  $a$  in the plane orthogonal to a unit vector  $n \in V$ . We have two-sided configuration constraints

$$C^0 = \{q \in Q; \langle g(q - q_0), n \rangle = 0, \|q - q_0\| = a\}, \quad (18)$$

$$C^1 = \mathbb{T}C^0 = \{(q, \delta q) \in \mathbb{T}Q; \langle g(q - q_0), n \rangle = 0, \|q - q_0\| = a, \langle g(\delta q), n \rangle = 0, \langle g(\delta q), u(q) \rangle = 0\}, \quad (19)$$

where  $u(q)$  is the unit vector  $(q - q_0)\|q - q_0\|^{-1}$ . The constitutive set

$$\begin{aligned} S &= \{(q, f) \in \mathbb{T}^*Q; q \in C^0, \langle f, \delta q \rangle = 0 \text{ for each } (q, \delta q) \in C^1\} \\ &= \{(q, f) \in \mathbb{T}^*Q; \langle g(q - q_0), n \rangle = 0, \|q - q_0\| = a, \langle f, u(q) \rangle = 0, \langle f, n \rangle = 0\} \end{aligned} \quad (20)$$

represents the statics of the system without friction and the constitutive set

$$\begin{aligned} S &= \{(q, f) \in \mathbb{T}^*Q; q \in C^0, \langle f, \delta q \rangle \leq \rho \|\delta q\| \text{ for each } (q, \delta q) \in C^1\} \\ &= \{(q, f) \in \mathbb{T}^*Q; \langle g(q - q_0), n \rangle = 0, \|q - q_0\| = a, \langle f, u(q) \rangle^2 + \langle f, n \rangle^2 \leq \rho^2\} \end{aligned} \quad (21)$$

takes constant friction into account. ▲

EXAMPLE 2. Let a material point be constrained to the exterior of a solid ball with the centre at  $q_0 \in Q$  and radius  $a$ . In its displacements on the surface of the ball the point encounters friction proportional to the component of the external force pressing the point against the surface. Correct representation of the statics of the point is obtained with one-sided constraints

$$C^0 = \{q \in Q; \|q - q_0\| \geq a\}, \quad (22)$$

$$C^1 = \left\{ (q, \delta q) \in \mathbb{T}Q; \|q - q_0\| \geq a, \langle g(\delta q), u(q) \rangle \geq \nu \sqrt{\|\delta q\|^2 - \langle g(\delta q), u(q) \rangle^2} \text{ if } \|q - q_0\| = a \right\} \quad (23)$$

and the constitutive set

$$\begin{aligned} S &= \{(q, f) \in \mathbb{T}^*Q; q \in C^0, \langle f, \delta q \rangle \leq 0 \text{ for each } (q, \delta q) \in C^1\} \\ &= \left\{ (q, f) \in \mathbb{T}^*Q; \|q - q_0\| \geq a, f = 0 \text{ if } \|q - q_0\| > a, \right. \\ &\quad \left. \nu \langle f, u(q) \rangle + \sqrt{\|f\|^2 - \langle f, u(q) \rangle^2} \leq 0 \text{ if } \|q - q_0\| = a \right\}, \end{aligned} \quad (24)$$

where  $u(q) = (q - q_0)\|q - q_0\|^{-1}$ . The constraints in this example are not configuration constraints. ▲

EXAMPLE 3. Let  $i$ ,  $j$ , and  $k$  be mutually orthogonal unit vectors and let  $q_0$  be a point. Let one-sided configuration constraints be specified by

$$C^0 = \{q \in Q; \langle g(q - q_0), i \rangle \geq 0, \langle g(q - q_0), j \rangle \geq 0\} \quad (25)$$

and

$$\begin{aligned} C^1 &= \mathsf{T}C^0 \\ &= \{(q, \delta q) \in \mathsf{T}Q; \langle g(q - q_0), i \rangle \geq 0, \langle g(q - q_0), j \rangle \geq 0, \\ &\quad \langle g(\delta q), i \rangle \geq 0, \text{ if } \langle g(q - q_0), i \rangle = 0, \langle g(\delta q), j \rangle \geq 0, \text{ if } \langle g(q - q_0), j \rangle = 0\}. \end{aligned} \quad (26)$$

The statics of a material point not subject to internal forces is represented by the constitutive set

$$\begin{aligned} S &= \{(q, f) \in \mathsf{T}^*Q; q \in C^0, \langle f, \delta q \rangle \leq \text{ for each } (q, \delta q) \in C^1\} \\ &= \{(q, f) \in \mathsf{T}^*Q; \langle g(q - q_0), i \rangle \geq 0, \langle g(q - q_0), j \rangle \geq 0, \langle f, k \rangle = 0, \\ &\quad \langle f, i \rangle = 0 \text{ and } \langle f, j \rangle \leq 0 \text{ if } \langle g(q - q_0), j \rangle = 0 \text{ and } \langle g(q - q_0), i \rangle \neq 0, \\ &\quad \langle f, j \rangle = 0 \text{ and } \langle f, i \rangle \leq 0 \text{ if } \langle g(q - q_0), i \rangle = 0 \text{ and } \langle g(q - q_0), j \rangle \neq 0, \\ &\quad \langle f, i \rangle \leq 0 \text{ and } \langle f, j \rangle \leq 0 \text{ if } \langle g(q - q_0), i \rangle = 0 \text{ and } \langle g(q - q_0), j \rangle = 0\} \end{aligned} \quad (27)$$

▲

EXAMPLE 4. In terms of the vectors  $i$ ,  $j$ , and  $k$  and the point  $q_0$  of the preceding example we define one-sided configuration constraints by

$$C^0 = \{q \in Q; \langle g(q - q_0), i \rangle \leq 0 \text{ or } \langle g(q - q_0), j \rangle \leq 0\} \quad (28)$$

and

$$\begin{aligned} C^1 &= \mathsf{T}C^0 \\ &= \{(q, \delta q) \in \mathsf{T}Q; \langle g(q - q_0), i \rangle \leq 0 \text{ or } \langle g(q - q_0), j \rangle \leq 0, \\ &\quad \langle g(\delta q), i \rangle \leq 0 \text{ if } \langle g(q - q_0), i \rangle = 0 \text{ and } \langle g(q - q_0), j \rangle \neq 0, \\ &\quad \langle g(\delta q), j \rangle \leq 0 \text{ if } \langle g(q - q_0), j \rangle = 0 \text{ and } \langle g(q - q_0), i \rangle \neq 0, \\ &\quad \langle g(\delta q), i \rangle \leq 0 \text{ or } \langle g(\delta q), j \rangle \leq 0 \text{ if } \langle g(q - q_0), i \rangle = 0 \text{ and } \langle g(q - q_0), j \rangle = 0\}. \end{aligned} \quad (29)$$

The statics of a material point not subject to internal forces is represented by the constitutive set

$$\begin{aligned} S &= \{(q, f) \in \mathsf{T}^*Q; q \in C^0, \langle f, \delta q \rangle \leq \text{ for each } (q, \delta q) \in C^1\} \\ &= \{(q, f) \in \mathsf{T}^*Q; \langle g(q - q_0), i \rangle \leq 0 \text{ or } \langle g(q - q_0), j \rangle \leq 0, \langle f, k \rangle = 0 \\ &\quad \langle f, i \rangle = 0 \text{ and } \langle f, j \rangle \geq 0 \text{ if } \langle g(q - q_0), j \rangle = 0 \text{ and } \langle g(q - q_0), i \rangle \neq 0, \\ &\quad \langle f, j \rangle = 0 \text{ and } \langle f, i \rangle \geq 0 \text{ if } \langle g(q - q_0), i \rangle = 0 \text{ and } \langle g(q - q_0), j \rangle \neq 0, \\ &\quad \langle f, i \rangle \geq 0 \text{ and } \langle f, j \rangle \geq 0 \text{ if } \langle g(q - q_0), i \rangle = 0 \text{ and } \langle g(q - q_0), j \rangle = 0\} \end{aligned} \quad (30)$$

▲

#### 4. Modeling configuration constraints in statics.

We believe that constraint static systems are idealized representations of unconstrained systems. The magnitude of the force that an external device can apply to a static system is limited and instruments used to observe displacements have a limited resolution. Idealizations take these limitations into account. We restrict the analysis to configuration constraints. A definition of configuration constraints will be based on the assumption that the norms of external forces at our disposal have an upper bound  $F$  and that displacements of distances less than  $d$  can not be detected.

Let  $C$  be a subset of  $Q$ . We denote by  $d(q, C)$  the distance of a point  $q \in Q$  from  $C$ . If  $C \subset Q$  is an embedded submanifold or a submanifold with smooth boundary, then for each configuration  $q$  sufficiently close to  $C$  there is a unique point  $q_C \in C$  nearest to  $q$ . The distance  $d(q, C) = \|q - q_C\|$  of  $q$  from  $C$  is a well defined function in a neighbourhood of  $C$ . If  $q$  is not in  $C$ , then  $q_C \neq q$  and the unit vector  $e(q) = (q - q_C)\|q - q_C\|^{-1}$  is orthogonal to  $C$  or the boundary of  $C$  at  $q_C$ .

Let  $C \subset Q$  be an embedded submanifold or a submanifold with smooth boundary and let  $S \subset \mathbb{T}^*Q$  be the constitutive set of a static system derived from the principle of virtual work

$$\langle f, \delta q \rangle \leq \sigma(q, \delta q) \text{ for each virtual displacement } (q, \delta q) \in \mathbb{T}C. \quad (31)$$

A model of this static system is constructed by choosing a function  $\bar{\sigma}$  on  $\mathbb{T}Q$  such that  $\sigma$  is the restriction of  $\bar{\sigma}$  to  $\mathbb{T}C$  and replacing the original principle of virtual work by the principle

$$\langle f, \delta q \rangle \leq \bar{\sigma}(q, \delta q) + kd(q, C)\langle g(e(q)), \delta q \rangle \text{ for each virtual displacement } (q, \delta q). \quad (32)$$

The term  $kd(q, C)\langle g(e(q)), \delta q \rangle$  is the directional derivative  $DK(q, \delta q)$  of the elastic internal energy function

$$K(q) = \frac{k}{2}(d(q, C))^2 \quad (33)$$

defined in the neighbourhood of  $C$  in which the distance function  $d(q, C)$  is well defined. The inequalities

$$kd(q, C) \leq \langle f, e(q) \rangle + \sigma(q, -e(q)) \quad (34)$$

and

$$kd(q, C) \leq |\langle f, e(q) \rangle| + |\sigma(q, -e(q))| \quad (35)$$

are derived from the principle of virtual work by setting  $\delta q = -e(q)$ . We will assume that  $F \ll kd$  and expect that the inequality  $|\sigma(q, -e(q))| \ll kd$  is satisfied. These inequalities together with  $|\langle f, e(q) \rangle| \leq \|f\| \leq F$  result in  $d(q, C) \ll d$ . It follows that using external forces at our disposal we can not induce the material point to assume configurations at noticeable distances away from  $C$ . It also follows that within the limits imposed by  $\|f\| \leq F$  the component  $\langle f, e(q) \rangle$  is arbitrary. Examples will be used to clarify details and present variations of this construction.

EXAMPLE 5. Let  $C$  be the set  $C^0$  of Example 1. We obtain the equation

$$(d(q, C))^2 = \|q - q_0\|^2 - 2a\sqrt{\|q - q_0\|^2 - \langle g(q - q_0), n \rangle^2} + a^2 \quad (36)$$

for the distance  $d(q, C)$  of  $q$  from  $C$  if this distance is less than  $a$ . If  $d(q, C) \neq 0$ , then

$$e(q) = \frac{q - q_0 - \langle g(q - q_0), n \rangle n}{\sqrt{\|q - q_0\|^2 - \langle g(q - q_0), n \rangle^2}} \quad (37)$$

is the unit vector orthogonal to  $C$  at the point  $q' \in C$  closest to  $q$  pointing from  $q'$  to  $q$ . Let a function  $\bar{\sigma}: \mathbb{T}Q \rightarrow \mathbb{R}$  be defined by  $\bar{\sigma}(q, n) = 0$ ,  $\bar{\sigma}(q, q - q_0) = 0$ , and  $\bar{\sigma}(q, \delta q) = \rho\|\delta q\|$  if  $\langle g(\delta q), n \rangle = 0$  and  $\langle g(\delta q), q - q_0 \rangle = 0$ . The unconstrained system represented by the principle of virtual work

$$\langle f, \delta q \rangle \leq \bar{\sigma}(q, \delta q) + kd(q, C)\langle g(e(q)), \delta q \rangle \text{ for each virtual displacement } (q, \delta q) \quad (38)$$

is a model of the constrained system of Example 1. It follows from the principle of virtual work that  $f = kd(q, C)g(e(q)) + f'$ , where the component  $f'$  satisfies relations  $\langle f', n \rangle = 0$ ,  $\langle f', q - q_0 \rangle = 0$ , and  $\|f'\| \leq \rho$ . If  $k \rightarrow \infty$ , then  $d(q, C) \rightarrow 0$ . Any value can be obtained for the component  $kd(q, C)g(e(q))$  as  $k \rightarrow \infty$  and  $d(q, C) \rightarrow 0$ . This is in agreement with the principle of virtual work of Example 1.  $\blacktriangle$

EXAMPLE 6. Let  $C$  be the set  $C^0$  of Example 2 and let  $u(q) = (q - q_0)\|q - q_0\|^{-1}$ . The distance  $d(q, C)$  is equal to  $\|q - q_0\| - a$ . A function  $\sigma: \mathbb{T}Q \rightarrow \mathbb{R}$  is defined by  $\sigma(q, \delta q) = 0$  if  $\|q - q_0\| \geq a$  and

$$\sigma(q, \delta q) = -kd(q, C)\langle g(u(q)), \delta q \rangle + kd(q, C)\nu\sqrt{\|\delta q\|^2 - \langle g(u(q)), \delta q \rangle^2} \quad (39)$$

if  $\|q - q_0\| < a$ . The principle of virtual work

$$\langle f, \delta q \rangle \leq \sigma(q, \delta q) \text{ for each virtual displacement } (q, \delta q) \in \mathbb{T}Q \quad (40)$$

implies the following relations for the external force  $f$ . If  $\|q - q_0\| \geq a$ , then  $f = 0$ . If  $\|q - q_0\| < a$ , then  $f = -kd(q, C)g(u(q)) + f'$  with  $\langle f', u(q) \rangle = 0$  and  $\langle f', \delta q \rangle \leq kd(q, C)\nu\|\delta q\|$  if  $\langle g(u(q)), \delta q \rangle = 0$ . If  $k \rightarrow \infty$ , then  $d(q, C) \rightarrow 0$ . The component  $\langle f, u(q) \rangle = -kd(q, C)$  can have any negative limit and  $\langle f', \delta q \rangle \leq -\langle f, u(q) \rangle\nu\|\delta q\|$ . This is in agreement with the principle of virtual work of Example 2.  $\blacktriangle$

EXAMPLE 7. A model for the system in Example 3 can be easily constructed even if the boundary of the set  $C = C^0$  is not smooth. The distance  $d(q, C)$  is defined by

$$d(q, C) = -\langle g(i), q - q_0 \rangle \quad (41)$$

if  $\langle g(j), q - q_0 \rangle \geq 0$  and  $\langle g(i), q - q_0 \rangle < 0$ ,

$$d(q, C) = -\langle g(j), q - q_0 \rangle \quad (42)$$

if  $\langle g(i), q - q_0 \rangle \geq 0$  and  $\langle g(j), q - q_0 \rangle < 0$ , and

$$d(q, C) = \sqrt{\langle g(i), q - q_0 \rangle^2 + \langle g(j), q - q_0 \rangle^2} \quad (43)$$

if  $\langle g(i), q - q_0 \rangle < 0$  and  $\langle g(j), q - q_0 \rangle < 0$ . A vector field  $e(q)$  is defined outside of  $C$  by  $e(q) = -i$  if  $\langle g(j), q - q_0 \rangle \geq 0$  and  $\langle g(i), q - q_0 \rangle < 0$ ,

$$e(q) = -j \quad (44)$$

if  $\langle g(i), q - q_0 \rangle \geq 0$  and  $\langle g(j), q - q_0 \rangle < 0$ , and

$$e(q) = (\langle g(i), q - q_0 \rangle i + \langle g(j), q - q_0 \rangle j)(d(q, C))^{-1} \quad (45)$$

if  $\langle g(i), q - q_0 \rangle < 0$  and  $\langle g(j), q - q_0 \rangle < 0$ . A function  $\sigma$  on  $\mathbb{T}Q$  is defined by  $\sigma(q, \delta q) = 0$  if  $q \in C$  and  $\sigma(q, \delta q) = k\langle g(e(q)), \delta q \rangle$  if  $q \notin C$ . The constitutive set of Example 3 is obtained from the principle of virtual work

$$\langle f, \delta q \rangle = \sigma(q, \delta q) \text{ for each virtual displacement } (q, \delta q) \in \mathbb{T}Q \quad (46)$$

with  $k \rightarrow \infty$  and  $d(q, C) \rightarrow 0$ .  $\blacktriangle$

EXAMPLE 8. The construction of the model in the preceding example followed exactly the prescription given at the beginning of the present section. This construction can not be directly applied to the set  $C = C^0$  of Example 4. It can be applied to the modified set

$$C_\varepsilon = C_0 \setminus \{q \in Q; \langle g(i), q - q_0 \rangle < r, \langle g(j), q - q_0 \rangle < r, \\ (\langle g(i), q - q_0 \rangle - r)^2 + (\langle g(j), q - q_0 \rangle - r)^2 > r\}. \quad (47)$$

The original set  $C_0$  is obtained as the limit as  $r \rightarrow 0$ .  $\blacktriangle$

## 5. Kinematics of autonomous systems and scleronomic constraints.

Motions of a material point in the Euclidean affine space  $Q$  are curves  $\xi: I \rightarrow Q$  parameterized by time  $t$  in an open interval  $I \subset \mathbb{R}$ . We have the *tangent prolongation*  $(\xi, \dot{\xi}): I \rightarrow \mathbb{T}Q: t \mapsto (\xi(t), \dot{\xi}(t))$  and the *second tangent prolongation*  $(\xi, \dot{\xi}, \ddot{\xi}): I \rightarrow \mathbb{T}^2Q: t \mapsto (\xi(t), \dot{\xi}(t), \ddot{\xi}(t))$  of a motion  $\xi$ .

Variational formulations of analytical mechanics require the concept of a *virtual displacement* of a motion. A virtual displacement of a motion  $\xi$  is a mapping  $(\xi, \delta\xi): I \rightarrow \mathbb{T}Q: t \mapsto (\xi(t), \delta\xi(t))$ . This mapping is obtained from a homotopy

$$\chi: \mathbb{R} \times I \rightarrow Q. \quad (48)$$

The base curve  $\chi(0, \cdot)$  is the motion  $\xi$ . The virtual displacement is the mapping

$$(\xi, \delta\xi): I \rightarrow \mathbb{T}Q: t \mapsto \mathbf{t}\chi(\cdot, t)(0). \quad (49)$$

A mapping  $(\xi, \dot{\xi}, \delta\xi, \delta\dot{\xi}): I \rightarrow \mathbb{TT}Q: t \mapsto (\xi(t), \dot{\xi}(t), \delta\xi(t), \delta\dot{\xi}(t))$  is obtained from a virtual displacement  $(\xi, \delta\xi)$  as the composition  $\kappa_Q \circ (\xi, \delta\xi, \dot{\xi}, \delta\dot{\xi})$  of the tangent prolongation  $(\xi, \delta\xi, \dot{\xi}, \delta\dot{\xi})$  with the involution  $\kappa_Q$ . Virtual displacements are subject to constraints. All considered versions of constraints can eventually be reduced to differential equations formulated in terms of a subset  $C^{(1,1)} \subset \mathbb{TT}Q$  such that if  $(q, \dot{q}, \delta q, \delta \dot{q}) \in C^{(1,1)}$ , then  $(q, \dot{q}, k\delta q, k\delta \dot{q}) \in C^{(1,1)}$  for each number  $k \geq 0$ . An *admissible virtual displacement*  $(\xi, \delta\xi)$  is required to satisfy the condition

$$(\xi(t), \dot{\xi}(t), \delta\xi(t), \delta\dot{\xi}(t)) \in C^{(1,1)} \quad (50)$$

for each  $t \in I$ . This condition implies conditions

$$(\xi(t), \dot{\xi}(t)) \in C^{(0,1)}, \quad (51)$$

$$(\xi(t), \delta\xi(t)) \in C^{(1,0)}, \quad (52)$$

and

$$\xi(t) \in C^{(0,0)} \quad (53)$$

for each  $t \in I$ . Sets  $C^{(0,1)}$ ,  $C^{(1,0)}$ , and  $C^{(0,0)}$  are defined by

$$C^{(0,1)} = \left\{ (q, \dot{q}) \in \mathbb{T}Q; (q, \dot{q}, \delta q, \delta \dot{q}) \in C^{(1,1)} \text{ for some } (\delta q, \delta \dot{q}) \in V \times V \right\}, \quad (54)$$

$$C^{(1,0)} = \left\{ (q, \delta q) \in \mathbb{T}Q; (q, \dot{q}, \delta q, \delta \dot{q}) \in C^{(1,1)} \text{ for some } (\dot{q}, \delta \dot{q}) \in V \times V \right\}, \quad (55)$$

and

$$C^{(0,0)} = \left\{ q \in Q; (q, \dot{q}) \in C^{(0,1)} \text{ for some } \dot{q} \in V \right\}. \quad (56)$$

Condition (51) is a differential equation for the motion  $\xi: I \rightarrow Q$ . Constraints are usually discussed in terms of this equation. The inclusion

$$C^{(0,1)} \subset \mathbb{T}C^{(0,0)} \quad (57)$$

must be satisfied since it is a necessary integrability condition for the equation (51). Constraints will be called *configuration constraints* if

$$C^{(0,1)} = \mathbb{T}C^{(0,0)}. \quad (58)$$

Velocity constraints are said to be *linear* if the set  $C^{(0,0)}$  is a submanifold of  $Q$  and  $C^{(0,1)}$  is a distribution on this submanifold. Linear constraints are said to be *holonomic* if  $C^{(0,1)}$  is integrable in the sense of Frobenius. Configuration constraints are a special case of holonomic constraints. Sets  $C^{(1,0)}$  and  $C^{(1,1)}$  are not usually discussed directly even if information contained in the *velocity constraints*  $C^{(0,1)}$  is not sufficient for the application of variational methods. The condition (50) is equivalent to

$$(\xi(t), \delta\xi(t), \dot{\xi}(t), \delta\dot{\xi}(t)) = \mathbf{t}(\xi, \delta\xi)(t) \in \kappa_Q(C^{(1,1)}) \subset \mathbb{TT}Q. \quad (59)$$

It is a differential equation for the virtual displacement  $(\xi, \delta\xi): I \rightarrow \mathbb{T}Q$ . The inclusion

$$C^{(1,1)} \subset \kappa_Q(\mathbb{T}C^{(1,0)}) \quad (60)$$

is a necessary integrability condition for this equation.

Two different methods of constructing the set  $C^{(1,1)}$  from the velocity constraints  $C^{(0,1)}$  are found in an article of Arnold, Kozlov, and Neishtadt [Arn].

- (1) In *vaconomic mechanics* the natural construction

$$C^{(1,1)} = \mathbb{T}C^{(0,1)} \quad (61)$$

is used. This construction is the result of the differential equation

$$\mathbf{t}\chi(s, \cdot)(t) \in C^{(0,1)} \quad (62)$$

imposed on curves  $\chi(s, \cdot)$  also with  $s \neq 0$ . See [Arn] for modifications of this construction necessary when virtual displacements vanishing at the ends of a time interval are required. With these modifications the formula (61) is still valid. The set  $C^{(1,0)}$  is the tangent set  $\mathbb{T}C^{(0,0)}$  of  $C^{(0,0)}$ .

- (2) The *d'Alembert-Lagrange* principle is based on the inclusion

$$C^{(1,1)} \subset \overline{C^{(1,1)}} = \left\{ (q, \dot{q}, \delta q, \delta \dot{q}) \in \mathbb{T}\mathbb{T}Q; (q, \dot{q}, 0, \delta q) \in \mathbb{T}C^{(0,1)} \right\}. \quad (63)$$

This construction derives from the condition

$$\mathbf{t}\chi(\cdot, t)(s) \in C^{(1,0)} \quad (64)$$

for each  $t \in I$  and each  $s$  and the condition

$$\mathbf{t}\chi(0, \cdot)(t) \in C^{(0,1)} \quad (65)$$

imposed on the base curve  $\xi = \chi(0, \cdot)$  but not the curves  $\chi(s, \cdot)$  for  $s \neq 0$ . The set  $\overline{C^{(1,1)}}$  may not represent an integrable differential equation (59) for  $(\xi, \delta\xi)$ . The set  $C^{(1,1)}$  is the integrable part of  $\overline{C^{(1,1)}}$ . If  $C^{(0,1)}$  is a vector subbundle of  $\mathbb{T}Q$ , then  $C^{(1,0)} = C^{(0,1)}$ . If  $C^{(0,1)}$  is an affine subbundle, then  $C^{(1,0)}$  is the model bundle. In both cases

$$\overline{C^{(1,1)}} = \left\{ (q, \dot{q}, \delta q, \delta \dot{q}) \in \mathbb{T}\mathbb{T}Q; (q, \dot{q}) \in C^{(0,1)}, (q, \delta q) \in C^{(1,0)} \right\} \quad (66)$$

and

$$C^{(1,1)} = \overline{C^{(1,1)}} \cap \mathbb{T}(\tau_{\mathbb{T}Q}(\overline{C^{(1,1)}})). \quad (67)$$

For configuration constraints both construction give the same result

$$C^{(1,1)} = \mathbb{T}\mathbb{T}C^{(0,0)}. \quad (68)$$

## 6. Dynamics of unconstrained autonomous systems.

We see four possible formulations of dynamics.

**A.** Dynamics of a material point can be specified as a collection of *boundary value relations with external forces* associated with time intervals. A boundary value relation for a time interval  $[a, b] \subset \mathbb{R}$  is a relation between an arc  $\xi: [a, b] \rightarrow Q$ , a mapping  $(\xi, \varphi): [a, b] \rightarrow \mathbb{T}^*Q$ , and two covectors  $(\xi(a), \pi(a))$  and  $(\xi(b), \pi(b))$ . The mapping  $(\xi, \varphi)$  represents the *external force* applied to the material point along the arc  $\xi$  the covectors  $(\xi(a), \pi(a))$ ,  $(\xi(b), \pi(b))$  are the *initial momentum* and *final momentum*. It



is convenient to consider the arc  $\xi: [a, b] \rightarrow Q$  and the mapping  $(\xi, \varphi): [a, b] \rightarrow \mathbb{T}^*Q$  the restrictions to the interval  $[a, b]$  of mappings  $\xi: I \rightarrow Q$  and  $(\xi, \varphi): I \rightarrow \mathbb{T}^*Q$  defined on an open interval  $I \subset \mathbb{R}$  containing  $[a, b]$ . The covectors  $(\xi(a), \pi(a))$  and  $(\xi(b), \pi(b))$  will be considered values of a mapping  $(\xi, \pi): I \rightarrow \mathbb{T}^*Q$  at the ends of the interval. The two mappings  $(\xi, \varphi): I \rightarrow \mathbb{T}^*Q$  and  $(\xi, \pi): I \rightarrow \mathbb{T}^*Q$  can be combined in a single mapping

$$(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*. \quad (69)$$

An element

$$((\xi, \varphi): [a, b] \rightarrow \mathbb{T}^*Q, (\xi(a), \pi(a)), (\xi(b), \pi(b))) \quad (70)$$

of the boundary value relation  $D_{[a,b]}$  for an interval  $[a, b]$  satisfies the *virtual action principle*

$$\begin{aligned} \int_a^b \langle \varphi(t), \delta \xi(t) \rangle dt - \langle \pi(b), \delta \xi(b) \rangle + \langle \pi(a), \delta \xi(a) \rangle \\ = \int_a^b \left( \lambda(\xi(t), \dot{\xi}(t), \delta \xi(t), \delta \dot{\xi}(t)) - m \langle g(\dot{\xi}(t)), \delta \dot{\xi}(t) \rangle \right) dt. \end{aligned} \quad (71)$$

for each virtual displacement  $(\xi, \delta \xi): [a, b] \rightarrow \mathbb{T}Q$  obtained as a restriction to  $[a, b]$  of a virtual displacement  $(\xi, \delta \xi): I \rightarrow \mathbb{T}Q$ . The term

$$m \langle g(\dot{\xi}(t)), \delta \dot{\xi}(t) \rangle \quad (72)$$

is the derivative  $DT(\xi(t), \dot{\xi}(t), \delta \xi(t), \delta \dot{\xi}(t))$  of the *kinetic energy* function

$$T: \mathbb{T}Q \rightarrow \mathbb{R}: (q, \dot{q}) \mapsto \frac{m}{2} \|\dot{q}\|^2 \quad (73)$$

of a material point with mass  $m$ . The function  $\lambda: \mathbb{T}\mathbb{T}Q \rightarrow \mathbb{R}$  represents the virtual action of internal forces. For the sake of simplicity we assume that  $\lambda$  is a linear form (a linear function of  $(\delta q, \delta \dot{q})$ ). The proposed principle of virtual action is more general than the Hamilton Principle. Note that the virtual displacements  $(\xi(a), \delta \xi(a))$  and  $(\xi(b), \delta \xi(b))$  of the end points of the arc do not vanish. Variational principles with variations of end points but without external forces were considered by Schwinger. The momentum-velocity relation is a law of physics and is a part of dynamics of a material point. This relation is included in the Schwinger version of the principle of virtual action but not in the Hamilton Principle.

Examples of the virtual action function  $\lambda$  include the function

$$\lambda: \mathbb{T}\mathbb{T}Q \rightarrow \mathbb{R}: (q, \dot{q}, \delta q, \delta \dot{q}) \mapsto e \langle A(q), \delta \dot{q} \rangle + e \langle DA(q, \delta q), \dot{q} \rangle \quad (74)$$

for a charged particle in a magnetic field derived from the vector potential  $A: Q \rightarrow V^*$  and the function

$$\lambda: \mathbb{T}\mathbb{T}Q \rightarrow \mathbb{R}: (q, \dot{q}, \delta q, \delta \dot{q}) \mapsto \gamma \langle g(\dot{q}), \delta q \rangle \quad (75)$$

for a material point immersed in a viscous medium. In the first of these examples the function  $\lambda(q, \dot{q}, \delta q, \delta \dot{q})$  is the derivative  $D\alpha(q, \dot{q}, \delta q, \delta \dot{q})$  of the function  $\alpha: \mathbb{T}Q \rightarrow \mathbb{R}: (q, \dot{q}) \mapsto e \langle A(q), \dot{q} \rangle$ . We will continue the analysis assuming that the function  $\lambda$  is of the simpler type

$$\lambda: \mathbb{T}\mathbb{T}Q \rightarrow \mathbb{R}: (q, \dot{q}, \delta q, \delta \dot{q}) \mapsto \langle \mu(q), \delta q \rangle, \quad (76)$$

where  $\mu$  is a mapping from  $\mathbb{T}Q$  to  $V^*$ . The principle of virtual action assumes the simpler form

$$\begin{aligned} \int_a^b \langle \varphi(t), \delta \xi(t) \rangle dt - \langle \pi(b), \delta \xi(b) \rangle + \langle \pi(a), \delta \xi(a) \rangle \\ = \int_a^b \left( \langle \mu(\xi(t)), \delta \xi(t) \rangle - m \langle g(\dot{\xi}(t)), \delta \dot{\xi}(t) \rangle \right) dt. \end{aligned} \quad (77)$$

Equivalent versions of this variational principle

$$\begin{aligned}
& \int_a^b \langle \varphi(t), \delta \xi(t) \rangle dt - \langle \pi(b), \delta \xi(b) \rangle + \langle \pi(a), \delta \xi(a) \rangle \\
&= \int_a^b \left( m \langle g(\ddot{\xi}(t)), \delta \xi(t) \rangle + \langle \mu(\xi(t)), \delta \xi(t) \rangle \right) dt \\
&\quad - m \langle g(\dot{\xi}(b)), \delta \xi(b) \rangle + m \langle g(\dot{\xi}(a)), \delta \xi(a) \rangle
\end{aligned} \tag{78}$$

and

$$\begin{aligned}
& \int_a^b \left( \langle \varphi(t) - \dot{\pi}(t), \delta \xi(t) \rangle - \langle \pi(t), \delta \dot{\xi}(t) \rangle \right) dt \\
&= \int_a^b \left( \langle \mu(\xi(t)), \delta \xi(t) \rangle - m \langle g(\dot{\xi}(t)), \delta \dot{\xi}(t) \rangle \right) dt
\end{aligned} \tag{79}$$

are easily derived by using the identities

$$\begin{aligned}
& \int_a^b \left( \langle \dot{\pi}(t), \delta \xi(t) \rangle + \langle \pi(t), \delta \dot{\xi}(t) \rangle \right) dt = \int_a^b \frac{d}{dt} \langle \pi(t), \delta \xi(t) \rangle dt \\
&= \langle \pi(b), \delta \xi(b) \rangle - \langle \pi(a), \delta \xi(a) \rangle
\end{aligned} \tag{80}$$

and

$$\begin{aligned}
& \int_a^b \left( m \langle g(\ddot{\xi}(t)), \delta \xi(t) \rangle + m \langle g(\dot{\xi}(t)), \delta \dot{\xi}(t) \rangle \right) dt = \int_a^b m \frac{d}{dt} \langle g(\dot{\xi}(t)), \delta \xi(t) \rangle dt \\
&= m \langle g(\dot{\xi}(b)), \delta \xi(b) \rangle - m \langle g(\dot{\xi}(a)), \delta \xi(a) \rangle.
\end{aligned} \tag{81}$$

---

**B.** Dynamics can be specified as the collection  $\mathcal{D}$  of curves

$$(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^* \tag{82}$$

defined on open intervals  $I \subset \mathbb{R}$  with the property that for each time interval  $[a, b] \subset I$  the arc  $(\xi, \varphi)|_{[a, b]}$  and the covectors  $(\xi(a), \pi(a))$  and  $(\xi(b), \pi(b))$  are in the boundary relation  $D_{[a, b]}$ .

---

**C.** Dynamics can be specified as differential equations

$$\varphi(t) - \dot{\pi}(t) = \mu(\xi(t)) \tag{83}$$

and

$$\pi(t) = g(\dot{\xi}(t)) \tag{84}$$

for mappings  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$ . These equations will be denoted by  $\dot{D}$ .

---

**D.** Dynamics can be specified as differential equations

$$\varphi(t) = mg(\ddot{\xi}(t)) + \mu(\xi(t)) \tag{85}$$

and

$$\pi(t) = g(\dot{\xi}(t)) \tag{86}$$

for mappings  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$ . These equations will be denoted by  $E$ .

Of the four formulations of dynamics version **A** is fundamental. The family of curves  $\mathcal{D}$  and the differential equations  $\dot{D}$  and  $E$  introduced in **B**, **C**, and **D** are auxiliary objects.

Differential equations  $\dot{D}$  and  $E$  are obviously equivalent.

We show that the family  $\mathcal{D}$  is the set of solutions of the equations  $\dot{D}$ . The proof is based on version (79) of the principle of virtual action. If a curve  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$  is in  $\mathcal{D}$  and  $(\xi, \delta\xi): I \rightarrow \mathbb{T}Q$  is an arbitrary virtual displacement, then the equality (79) holds for all intervals  $[a, b] \subset I$ . It follows that the equality

$$\langle \varphi(t) - \dot{\pi}(t), \delta\xi(t) \rangle - \langle \pi(t), \delta\dot{\xi}(t) \rangle = \langle \mu(\xi(t)), \delta\xi(t) \rangle - m \langle g(\dot{\xi}(t)), \delta\dot{\xi}(t) \rangle \quad (87)$$

holds at each  $t \in I$ . This implies that equations  $\dot{D}$  are satisfied due to arbitrariness of the vectors  $\delta\xi(t)$  and  $\delta\dot{\xi}(t)$ . Conversely if  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$  is a solution of  $\dot{D}$ , then the equality (87) holds in  $I$  with an arbitrary displacement  $(\xi, \delta\xi): I \rightarrow \mathbb{T}Q$ . The validity of the principle of virtual action for each time interval  $[a, b] \subset I$  is established by integration. Hence,  $(\xi, \varphi, \pi)$  is in  $\mathcal{D}$ .

It follows from the definition of  $\mathcal{D}$  that this family is constructed from the boundary value relations. We show that elements of boundary value relations can be constructed from elements of  $\mathcal{D}$ . Let  $[a, b]$  be a time interval included in an open interval  $I \subset \mathbb{R}$  and let  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$  be a mapping such that  $(\xi, \varphi)|_{[a, b]}, (\xi(a), \pi(a)), (\xi(b), \pi(b))$  is in  $D_{[a, b]}$ . It follows from version (78) of the principle of virtual action that the mapping  $(\xi, \varphi)$  satisfies the equation

$$\varphi(t) = mg(\ddot{\xi}(t)) + \mu(\xi(t)) \quad (88)$$

in  $[a, b]$  and that

$$\pi(a) = g(\dot{\xi}(a)) \quad \text{and} \quad \pi(b) = g(\dot{\xi}(b)). \quad (89)$$

Let mappings  $\varphi': I \rightarrow V^*$  and  $\pi': I \rightarrow V^*$  be defined by

$$\varphi'(t) = mg(\ddot{\xi}(t)) + \mu(\xi(t)) \quad (90)$$

and

$$\pi'(t) = mg(\dot{\xi}(t)). \quad (91)$$

The mapping  $(\xi, \varphi', \pi'): I \rightarrow Q \times V^* \times V^*$  is in  $\mathcal{D}$  since it is a solution of  $E$ . The boundary value data extracted from this mapping are in the boundary value relation since  $(\xi, \varphi')|_{[a, b]} = (\xi, \varphi)|_{[a, b]}$ ,  $\pi'(a) = \pi(a)$ , and  $\pi'(b) = \pi(b)$ .

## 7. Dynamics with configuration constraints.

We list four possible formulations of dynamics with constraints analogous to the four formulations in the preceding section. As defined in Section 5 an admissible virtual displacement is a mapping  $(\xi, \delta\xi): I \rightarrow \mathbb{T}Q$  satisfying the condition

$$(\xi(t), \dot{\xi}(t), \delta\xi(t), \delta\dot{\xi}(t)) \in C^{(1,1)} = \mathbb{T}\mathbb{T}C^{(0,0)} \quad (92)$$

for each  $t \in I$ .

**A.** Dynamics of a material point can be considered a collection of *boundary value relations* associated with time intervals. An element

$$((\xi, \varphi): [a, b] \rightarrow \mathbb{T}^*Q, (\xi(a), \pi(a)), (\xi(b), \pi(b))) \quad (93)$$

of the boundary value relation  $D_{[a, b]}$  for an interval  $[a, b]$  satisfies the *virtual action principle*

$$\begin{aligned} \int_a^b \langle \varphi(t), \delta\xi(t) \rangle dt - \langle \pi(b), \delta\xi(b) \rangle + \langle \pi(a), \delta\xi(a) \rangle \\ = \int_a^b \left( \langle \mu(\xi(t)), \delta\xi(t) \rangle - m \langle g(\dot{\xi}(t)), \delta\dot{\xi}(t) \rangle \right) dt. \end{aligned} \quad (94)$$

for each admissible virtual displacement  $(\xi, \delta\xi): [a, b] \rightarrow \mathbb{T}Q$  obtained as a restriction to  $[a, b]$  of an admissible virtual displacement  $(\xi, \delta\xi): I \rightarrow \mathbb{T}Q$ . The mapping  $\mu$  is defined on  $C^{(1,0)} = \mathbb{T}C^{(0,0)}$ . The condition  $\xi(t) \in C^{(0,0)}$  for each  $t \in I$  is implied.

There are again the equivalent versions of this variational principle

$$\begin{aligned} & \int_a^b \langle \varphi(t), \delta\xi(t) \rangle dt - \langle \pi(b), \delta\xi(b) \rangle + \langle \pi(a), \delta\xi(a) \rangle \\ &= \int_a^b \left( m \langle g(\ddot{\xi}(t)), \delta\xi(t) \rangle + \langle \mu(\xi(t)), \delta\xi(t) \rangle \right) dt \\ & \quad - m \langle g(\dot{\xi}(b)), \delta\xi(b) \rangle + m \langle g(\dot{\xi}(a)), \delta\xi(a) \rangle \end{aligned} \quad (95)$$

and

$$\begin{aligned} & \int_a^b \left( \langle \varphi(t) - \dot{\pi}(t), \delta\xi(t) \rangle - \langle \pi(t), \delta\dot{\xi}(t) \rangle \right) dt \\ &= \int_a^b \left( \langle \mu(\xi(t)), \delta\xi(t) \rangle - m \langle g(\dot{\xi}(t)), \delta\dot{\xi}(t) \rangle \right) dt. \end{aligned} \quad (96)$$

**B.** Dynamics can be specified as the collection  $\mathcal{D}$  of curves

$$(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^* \quad (97)$$

defined on open intervals  $I \subset \mathbb{R}$  such that for each time interval  $[a, b] \subset I$  the arc  $(\xi, \varphi)|_{[a, b]}$  and the covectors  $(\xi(a), \pi(a))$  and  $(\xi(b), \pi(b))$  are in the boundary relation  $D_{[a, b]}$ .

**C.** Dynamics can be specified as the differential equation

$$\langle \varphi(t) - \dot{\pi}(t), \delta\xi(t) \rangle - \langle \pi(t), \delta\dot{\xi}(t) \rangle = \langle \mu(\xi(t)), \delta\xi(t) \rangle - \langle g(\dot{\xi}(t)), \delta\dot{\xi}(t) \rangle \quad (98)$$

to be satisfied by a curve  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$  at each  $t \in I$  and for each  $(q, \dot{\xi}(t), \delta\xi(t), \delta\dot{\xi}(t)) \in C^{(1,1)}$ . This is equivalent to equations

$$\langle \varphi(t) - \dot{\pi}(t), \delta\xi(t) \rangle = \langle \mu(\xi(t)), \delta\xi(t) \rangle \quad (99)$$

and

$$\langle \pi(t), \delta\xi(t) \rangle = \langle g(\dot{\xi}(t)), \delta\xi(t) \rangle \quad (100)$$

satisfied at each  $t \in I$  for each  $(\xi(t), \delta\xi(t)) \in C^{(1,0)}$ .

**D.** Dynamics can be specified as the differential equations

$$\langle \varphi(t), \delta\xi(t) \rangle = \langle mg(\ddot{\xi}(t)) + \mu(\xi(t)), \delta\xi(t) \rangle \quad (101)$$

and

$$\langle \pi(t), \delta\xi(t) \rangle = \langle g(\dot{\xi}(t)), \delta\xi(t) \rangle \quad (102)$$

satisfied by a mapping  $(\xi, \varphi, \pi): I \rightarrow Q \times V^* \times V^*$  at each  $t \in I$  and each  $(\xi(t), \delta\xi(t)) \in C^{(1,0)}$ .

The four formulations are valid for configuration constraints and are equivalent as in the case of unconstrained systems. The situation is much more complex in the case of more general constraints. The method of models could be a tool for testing the validity of different formulations. We will apply this tool to the momentum-velocity relation. Note that the usual momentum-velocity relation

$$\pi(t) = g(\dot{\xi}(t)) \quad (103)$$

is replaced by the equation (102). We will attempt a justification of this modification of the momentum-velocity relation based on models of configuration constraints.

## 8. Models of autonomous systems with configuration constraints.

Let a material point of mass  $m$  be constrained to a plane  $C^{(0,0)} \subset Q$  passing through a point  $q_0$  and orthogonal to a unit vector  $n$ . We will assume that there are no internal forces and no external forces are applied. The constraint will be modeled by a strong internal elastic force  $k\langle g(q - q_0), n \rangle g(n)$ . Let an initial momentum  $(\xi(a), \pi(a))$  such that  $\xi(a) \in C^{(0,0)}$  and  $\langle \pi(a), n \rangle = 0$  be applied to the point. The solution of the dynamical equations will be the mapping  $(\xi, \varphi, \pi): R \rightarrow Q \times V^* \times V^*$  with  $\xi(t) = \xi(a) + m^{-1}g^{-1}(\pi(a))(t - a)$ ,  $\varphi(t) = 0$ , and  $\pi(t) = \pi(a)$ . If the constraint is replaced by the elastic force the solution mapping will be the same. Let now the initial momentum have a non zero component  $\langle \pi(a), n \rangle$ . For the unconstrained model the solution is the mapping  $(\xi, \varphi, \pi)$  with

$$\xi(t) = \xi(a) + m^{-1} (g^{-1}(\pi(a)) - \langle \pi(a), n \rangle n) (t - a) + \omega^{-1} \langle \pi(a), n \rangle n \sin \omega(t - a), \quad (104)$$

$$\varphi(t) = 0,$$

$$\pi(t) = \pi(a) + \langle \pi(a), n \rangle g(n) (\cos \omega(t - a) - 1), \quad (105)$$

and  $\omega = \sqrt{k/m}$ . The oscillation may be invisible since the amplitude  $\omega^{-1} \langle \pi(a), n \rangle$  may be small due to the high value of  $\omega$ . The rapidly changing component  $\langle \pi(a), n \rangle g(n) \cos \omega(t - a)$  of the momentum transverse to the plane  $C^{(0,0)}$  depends on the initial value  $\langle \pi(a), n \rangle g(n)$  and is arbitrary within certain limits. This component can be detected by making the material point collide with an unconstrained mass. Time dependent external forces and curvature of constraint set  $C^{(0,0)}$  may even cause the transverse component of momentum influence the visible part of the motion along the constraint. The element of the boundary value relation for the idealized constrained system is composed of the mapping  $(\xi, \varphi): [a, b] \rightarrow Q \times V^*$  with  $\xi(t) = \xi(a) + m^{-1} (g^{-1}(\pi(a)) - \langle \pi(a), n \rangle n) (t - a)$ ,  $\varphi(t) = 0$ , and covectors  $(\xi(a), \pi(a))$  and  $(\xi(b), \pi(b))$  satisfying the equality  $\pi(b) - \langle \pi(b), n \rangle g(n) = \pi(a) - \langle \pi(a), n \rangle g(n)$ . The transverse component  $\langle \pi(a), n \rangle g(n)$  of the initial momentum is arbitrary. Due to the rapidity of oscillations the final value of the transverse component  $\langle \pi(b), n \rangle g(n)$  of final momentum should be considered arbitrary and independent of the initial value.

This analysis based on a purely elastic model suggests that the virtual action principle

$$\begin{aligned} & \int_a^b \langle \varphi(t), \delta \xi(t) \rangle dt - \langle \pi(b), \delta \xi(b) \rangle + \langle \pi(a), \delta \xi(a) \rangle \\ &= \int_a^b \left( m \langle g(\ddot{\xi}(t)), \delta \xi(t) \rangle + \langle \mu(\xi(t)), \delta \xi(t) \rangle \right) dt \\ & \quad - m \langle g(\dot{\xi}(b)), \delta \xi(b) \rangle + m \langle g(\dot{\xi}(a)), \delta \xi(a) \rangle \end{aligned} \quad (106)$$

for each admissible virtual displacement  $(\xi, \delta \xi): [a, b] \rightarrow TQ$  is appropriate for material points subject to configuration constraints. If this formulation of dynamics with configuration constraints is adopted, then the momentum-velocity relation

$$\text{momentum} = \text{mass} \times \text{velocity}$$

is no longer valid. Velocity is the rate of change of configuration. The component of velocity transverse to the constraint set is zero. This is not true of the transverse component of momentum. Along directions tangent to the constraint the usual momentum-velocity relation holds. Other models may be found appropriate in certain situations. When a rigid bar is struck with a hammer it starts an invisible vibration detectable through the sound it emits. The sound is due to momentum (and energy) transfer to air molecules colliding with the surface of the bar. The vibration will eventually die away although it may continue for a long time if the bar is placed in vacuum. The inevitable damping can be taken into account by supplementing the elastic force  $k\langle g(q - q_0), n \rangle g(n)$  with a viscous damping force  $\gamma \langle g(\dot{\xi}(t)), n \rangle g(n)$ . It may be appropriate to consider the transverse component of the initial momentum arbitrary and the transverse component of the final momentum effectively reduced to zero. In cases of relatively short time intervals the effect of the damping can be ignored. In cases of strong damping and relatively long time intervals. It may be correct to assume that the transverse component of the initial momentum is completely absorbed in an essentially inelastic collision with the suspension and

not transferred to the material point. In such cases the boundary value relation will be a solution of the Hamilton principle with external forces:

$$\int_a^b \langle \varphi(t), \delta \xi(t) \rangle dt = \int_a^b \left( m \langle g(\ddot{\xi}(t)), \delta \xi(t) \rangle + \langle \mu(\xi(t)), \delta \xi(t) \rangle \right) dt \quad (107)$$

for each admissible virtual displacement  $(\xi, \delta \xi): [a, b] \rightarrow \mathbb{T}Q$  with  $\delta \xi(a) = 0$  and  $\delta \xi(b) = 0$ . The equalities  $\pi(a) = g(\dot{\xi}(a))$  and  $\pi(b) = g(\dot{\xi}(b))$  supplement the variational principle.

## 9. References.

[Arn] V.I. Arnold, V.V. Kozlov, and A.I. Neishtadt, Mathematical Aspects of Classical and Celestial Mechanics in *Dynamical Systems III*, V.I. Arnold (Ed.), Springer-Verlag